

Relativity free of coordinates

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Abstract

The concept of an inertial reference frame, which actualizes Euclidean geometry, is not confined to the statics of hardly deformable solids but extendible to the dynamical phenomena where Newtonian mechanics is valid, defining its concept of time. The laws of propagation of electromagnetic disturbances modify Newtonian formalism for sufficiently fast free motions within each spatial domain of its validity for slow motions and introduce the extended concept of time by uniting those of Newtonian which can exist in different spatial domains of their validity.

A boost direction for a pair of inertial reference frames is that spatial direction in one of the frames along which the other frame moves. Free motions of point particles make an instrumentation for identifying the boost direction as well as events on a straight line along that direction. The concept of a boost direction secures the physics-based formulation of the basic relativity effects: the time dilation and retardation, the contraction of the length along and the spatial invariance across the direction of relative motion of two frames.

Eventually, that formulation results in the relation between two arbitrary frames in terms of their position vectors and time moments for a given event. The obtained transformation rules for the components of the position vector differ from the vector-like relationship known in the literature because the latter actually deals with column vectors made of Cartesian coordinates of true vectors and appears identical to a boost coordinate transformation.

Within the physics-based approach, addressing the transformation of coordinates implies specifying some coordinate systems in each frame in terms of physical objects/directions. This yields a logically consistent and physically meaningful presentation of the coordinate transformations commonly exploited in the special relativity theory, which makes observable effects associated with those transformations evident. In particular, for a Cartesian coordinate system subjected to a boost coordinate transformation, the coordinate-free technique of reasoning allows one to evaluate its instantaneously observable apparent distortion easily.

I. MOTIVATION

The safely correct development and credible presentation of a physical theory implies identifying physical phenomena to which the theory is assuredly applicable, however small such a validity domain would seem to be at the outset.

Historically, the special relativity theory came along in attempts to make Newtonian/Euclidean description of moving macroscopic bodies consistent with optical phenomena observable by means of these bodies. Einstein's seminal paper¹ establishes the principle of relativity and an idea that propagation of light does not depend on the motion of its source and can be used to define simultaneous events at different places as premises of a physically reasonable way to arrive at Lorentz transformation, reproduced in subsequent monographs² and textbooks⁸.

The alternative post-Einsteinian presentation¹⁴ of the relativity theory brought Einstein's attempts to identify physical foundations of the problem to its actual formal source: the required change of a coordinate system should preserve the form of free motion of point particles along with the speed of light c as a universal limitation on the speed of any particle. Such a mathematical approach appears physically important because it provides a consistent description of the macroscopically perceived (commonly referred to as kinematic¹⁵) part of experimental particle physics: Registering devices embedded in the solid wall of an accelerator can embody an inertial reference frame along with its Euclidean geometry; registrable/inferable collisions between particles are events.

Up to the present time, all expositions of the special relativity theory heavily rely on Cartesian coordinates for both formulating premises and developing inferences. Apparently, within macroscopic physics resorting to that handy mathematical technique imposes no restriction on the applicability of such a reasoning. However, good practice of using coordinates in physics and engineering implies that the choice of coordinate systems should facilitate the application of basic/general regularities/rules to a particular problem, so that it is the spatial configuration of the problem that actually determines whether a suitable coordinate system is Cartesian, or orthogonal curvilinear, or even special such as barycentric etc.

In the next section the reader can find a remainder of what physics secures the existence of frames and actually underlies the validity of Euclidean geometry. Section III presents

the concept of a boost direction, which enables the subsequent physics oriented inference of the well known spatio-temporal effects related to the special relativity theory. Section IV exploits these effects to formulate the relation between the time moments and the position vectors of a given event in two frames. This relation entails a logically consistent and physically meaningful presentation of the well known coordinate transformations in Section V.

II. INERTIAL FRAMES

A. Euclidean geometry and Newtonian mechanics

Within macroscopic physics, a (necessarily inertial) frame can be provisionally viewed as a rigid construct, which means that the spatial relations between its parts obey the rules of Euclidean geometry. Making use of this well known mathematical structure in the description of undeformed (or, equivalently, non-accelerating and non-rotating) macroscopic solids is evidently valid, which suggests that the regularities of statics (supplemented with Hooke law to resolve statics' ambiguities) can underlie Euclidean geometry. Usually, this formal struture is believed to be applicable to a wider range of spatial scales as well as a larger class of physical phenomena. So other regularities, such as those of electromagnetism and gravity, may take part in maintaining Euclidean geometry for the set of all possible positions of sufficiently small interacting bodies.¹⁶

As of now, the group of motions of rigid bodies (described axiomatically) is the only mathematical structure that reminds us of the physical foundation of Euclidean geometry. The group includes spatial translations \hat{T} and rotations \hat{R} . The additive representation of a spatial translation is usually referred to as a spatial vector. One can use rotations to introduce an angle between two vectors etc. When the orthonormal vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ represent the translations along three mutually perpendicular directions, the decomposition

$$\Delta \mathbf{r} = \Delta x \mathbf{e}_x + \Delta y \mathbf{e}_y + \Delta z \mathbf{e}_z \quad (1)$$

of a displacement (the change of a position vector \mathbf{r}) is just what defines Cartesian coordinates. This formal structure refers those interested in the origin of Euclidean geometry to the physics of solids, which may not be a reasonable starting point of an investigation aimed something in the physics of fields and particles.

Meanwhile, it is Newtonian laws that make an established formalism of mechanics. Within the limitations imposed by microscopic phenomena, Newtonian formalism is believed to be applicable to each sufficiently small part of a macroscopic body, referred to as a point particle.

To calculate how the position vector \mathbf{r}_a of each particle a is changing over time, a theorist should invoke Newton's second law

$$\left\{ m_a \frac{d^2 \mathbf{r}_a}{dt^2} = \sum_b \mathbf{F}_{ab} \right\}. \quad (2)$$

Here and hereinafter the notation $\{g_a\}$ is used for the list of expressions g_a where the label a runs over all its values.

Within the purely mechanical formalism, the force of action of a particle b on a particle a cannot be but conservative:

$$\mathbf{F}_{ab} = -\nabla_a U(|\mathbf{r}_a - \mathbf{r}_b|),$$

where ∇_a denotes the nabla operator designed to act on functions of the position vector \mathbf{r}_a .

For an isolated system of N particles and given functions $\{U(|\mathbf{r}_a - \mathbf{r}_b|)\}$, the decomposition (1) turns Eqs. (2) into a closed set of $3N$ differential equations of second order. Its general solution

$$\{\mathbf{r}_a = \mathbf{r}_a(t + \tau; E, \mathbf{P}, \mathbf{L}, \{I_{\mathcal{A}}\})\} \quad (3)$$

involves $6N$ arbitrary constants, of which one is the time reference shift τ while seven can be the total energy E and the components of the total momentum \mathbf{P} and the total angular momentum \mathbf{L} . In other words, inversion of the set of Eqs. (3) can yield $6N-8$ constants of motions $\{I_{\mathcal{A}} = I_{\mathcal{A}}(\{\mathbf{r}_a\}, \{\dot{\mathbf{r}}_a\}, t)\}$, possibly specific for each sufficiently small interval of t , in addition to the seven universal constants of motion:

$$E = \frac{1}{2} \sum_a m_a \dot{\mathbf{r}}_a^2 + \sum_{a \neq b} U(|\mathbf{r}_b - \mathbf{r}_a|),$$

$$\mathbf{P} = \sum_a m_a \dot{\mathbf{r}}_a,$$

$$\mathbf{L} = \sum_a m_a [\mathbf{r}_a \times \dot{\mathbf{r}}_a].$$

Of these functions, only $\{I_{\mathcal{A}}\}$ essentially represent the internal/relative motions of the particles while E describes the overall intensity of the whole motion, which can be arbitrary

normalized by the choice of the unit of the time t ; \mathbf{P} and \mathbf{L} correspond to the well known global directions related to the whole motion.

The formalism based on the use of position vectors, their decompositions (1) and Eqs. (2) may not be a consistent introduction to foundations since it refers to the regularities/notions left without description in terms of the physics phenomena involved. To build a foundational construction straightforwardly, one should identify some real or possible stationary objects with Euclidean points so that the appropriate constants of motion can approximate the values of the one-point vector field $\{\mathbf{e}_\alpha\}$ in Eq. (1), two-point scalar field known as distance (Euclidean length) etc.

The well-known manifestations of quantization preclude implementing such a construction at sufficiently small scales.¹⁸ As a result, nor Eqs. (2) nor Eqs. (3) appear applicable at those scales. Nevertheless, to construct Euclidean geometry (or, at least, some pregeometry) one could still adopt the motions of the interacting particles as primitive notions described with a set of experimentally identifiable relations between them. The formulation of such relations (which would then make a low-level automatics of mechanics) is far from the goals of this text, but there is hardly any doubt that the set of motions of interacting particles is sufficiently rich to support Euclidean geometry and, therefore, the concept of a frame. At this logical level, the relations that underlie Euclidean geometry are not separable from those that eventually give rise to the existence of the constants of motion.

B. Inertial frames and relativity principle

As discussed in the previous section, within the familiar high-level formalism the concept of a frame manifests itself by means of a position vector \mathbf{r} , Euclidean nature of which is most likely secured by non-relativistic classical physics. At least one connection between frames and regularities of classical physics reveals itself in terms of a position vector: Eqs. (2) keep their form when one changes a frame B for a frame A in accordance with the transformation rule

$$\left. \begin{aligned} \mathbf{r}^{(B)} &= \mathbf{r}^{(A)} + \mathbf{a} - \mathbf{v}_B^{(A)} t^{(A)}, \\ t^{(B)} &= t^{(A)} + \tau; \end{aligned} \right\} \quad (4)$$

where \mathbf{a} , $\mathbf{v}_B^{(A)}$ and τ are given parameters. Here and hereinafter the superscript (F) indicates that a quantity $q^{(F)}$ is initially defined in a frame F. (But as far as the transformation (4) is

valid between any pair of frames, one can actually define a position vector \mathbf{r} in any frame.)

From the early days of the relativity theory, the following generalization of the above statement is regarded as a more or less universal principle, called the principle of relativity: The mutual disposition and the relative translational uniform rectilinear motion of two frames cannot manifest itself in the description of physical phenomena within one of these frames. Equivalently, physical laws have the same formulations in different (necessarily non-accelerated non-rotating) frames.²²

In the context of Newtonian mechanics, the meaning of the relativity principle is plain: translational uniform rectilinear motion of a physical system as a whole with respect to some external (reference) bodies does not affect the motion of the internal parts of the system with respect to each other. However, this idea presumes a partition of the physical system that cannot be seamlessly extended to include electromagnetic phenomena since the decomposition of an electromagnetic field into non-interfering components is possible only without electric charges. This apparent gap is accompanied (and aggravated) by the fact that the transformation (4) does not preserve the form of source-free Maxwell's equations.

Naturally, an attempt to derive the general transformation that keeps the full formalism of Maxwellian electrodynamics would lead a researcher to a complicated problem. So the developers of the special relativity theory cannot but begin with the simple coordinate transformation known as (the original form of) Lorentz transformation (which, in accordance with the purpose and logic of this article, is explicitly reproduced as late as in Section V A.)

Looking into early presentations of the special relativity theory, one can identify the following physics premise for the formal derivation of Lorentz transformation: the law of motion of a free particle and the laws of propagation of a free electromagnetic field have the same form in all frames.²⁵ To be exact, in an arbitrary frame F the position $\mathbf{r}_a^{(F)}$ of a free particle is changing along with the time $t^{(F)}$ as

$$\mathbf{r}_a^{(F)} = \mathbf{r}_{a0}^{(F)} + \mathbf{v}_a^{(F)} t^{(F)} \quad (5)$$

while the positions $\mathbf{r}^{(F)}$ taken by an electromagnetic wavefront (wave phase-front) at the time $t^{(F)}$ from a point source flashed at the position $\mathbf{r}_0^{(F)}$ at the time $t_0^{(F)}$ satisfy

$$\left(\mathbf{r}^{(F)} - \mathbf{r}_0^{(F)}\right)^2 = c^2 \left(t^{(F)} - t_0^{(F)}\right)^2. \quad (6)$$

The propagation speed c of an electromagnetic spherical wavefront is the same in all frames. As long as the goals of one's inference are limited by the derivation of rules equivalent to

those of Lorentz transformation, one can confine oneself with the limiting form of Eq. (6) for an infinitely far source position, i.e. the equation

$$(\mathbf{n}^{(F)} \cdot \mathbf{r}^{(F)}) - ct^{(F)} = \text{const} \quad (7)$$

for a plane wavefront which propagates in the direction of the unit vector $\mathbf{n}^{(F)}$.

One can view the above statements as a partial realization of Einstein's original intention to extend the principle of relativity to electromagnetic phenomena. Meanwhile, the next generation of authors has dispensed with both Maxwell equations and Eq. (6) in their introductions to the relativity theory. In the post-Einsteinian derivations of the transformation rules between two frames one finds the principle of relativity replaced by the requirement to preserve the form of Eq. (5) supplemented with the condition

$$(\mathbf{v}^{(F)} \cdot \mathbf{v}^{(F)}) = c^2 \quad (8)$$

to include "the motion of a light signal" (in effect, the propagation of the intersection point of an electromagnetic plane wavefront with one of its associated ray paths.)²⁶ As a result, the physics that underlies Einstein's special relativity theory has been reduced to that of free point particles with the universal limitation on their speed, equal to c in all frames.

The use of free particles in a reasoning is evidently restricted by the processes of particles' interaction. But as long as one neglects the extension and the duration of those processes, one can exploit an interaction act as a representation of a basic identifiable entity usually referred to as an event. In other words, each event appears to be real or possible interception of two (or more) free particles. Then one can exploit experimentally identifiable relations between motions of free particles to establish relations between events. If need, the interaction between particles can be assumed so small as not to change their motion. (In general, one need exploiting Eq. (5) for the motion of each particle a between the points of interception, where the parameters $\mathbf{v}_a^{(F)}$ may change.) This technique along with the basic appliance of the relativity principle is a main tool used in the next section.

Here it is also worth noting that the free motion of particles and the propagation of light rays are not sufficient to construct Euclidean geometry, so the attempts to extend its validity to arbitrarily fast processes could end up with nothing but a new postulate.²⁷ In order to have material carriers of Euclidean geometry, presentations of the relativity theory have no choice but to borrow frames from Newtonian mechanics.

When someone applies Newton's second law (2) to a physical system which consists of weakly interacting (e.g., widely separated) parts, he might think that Newtonian mechanics should involve some means to identify motions in such parts as simultaneous processes. Actually, Eqs. (2) are well known to have originated from the regularities revealed by experiments/observations related to strongly interacting physical bodies, especially gravitationally bound ones, such as Sun and planets. But in such a system, the existence of the time t is an inherent property of its (almost periodic or quasi-periodic) motion. If v is a characteristic speed of such motion, then for a given timescale Δt it secures synchronization in changing physical quantities over the region of size $l \sim v\Delta t$. However, the special relativity theory implies the speed of light c as a characteristic speed and, therefore, considers the region of size $L \sim c\Delta t \gg l$. In other words, within the special relativity theory, Newtonian mechanics can secure the synchronization two processes ("clocks") but only at one spatial point.

Exploiting Eq. (6)/Eq. (7) or the light rays only (without specifying geometrical structures prematurely) one can synchronize events happened to particles at different positions (as far as one neglects the time delay and the position shift due to interaction of a charged particle and electromagnetic field.) This turns the time moment t into a global variable, similar to a spatial coordinate.

In principle, within sufficiently small timescales, Newtonian mechanics can maintain Euclidean geometry only in the vicinity of each event, which means that the geometry of the subset $t = \text{const}$ might be Riemannian.³⁰ In this text, the global geometry of particles' positions within each frame is still assumed Euclidean, since it is appropriate for usual practical applications of the special relativity theory as well as common teaching curricula.

III. BASIC EFFECTS OF THE RELATIVITY THEORY

A. Boost direction

For any pair of frames A and B, there are the velocity $\mathbf{v}_A^{(B)}$ of A with respect to B and, vice versa, the velocity $\mathbf{v}_B^{(A)}$. Since each of these vectors is defined as a spatial object in its own frame, there can be no procedure of comparing them directly on the basis of Newtonian mechanics. Nevertheless, since all frames are supposed to be identical in their essential

internal properties, one should accept

$$|\mathbf{v}_A^{(B)}| = |\mathbf{v}_B^{(A)}| \equiv v_{AB} \equiv v \quad \gamma_{AB} \equiv \gamma(v_{AB}) = \gamma(v) \equiv \gamma \quad (9)$$

due to the symmetry of exchanging A and B. (This should be also considered as a part of establishing universal time unit since one cannot be sure that a nonrelativistic standard clocks keeps its rate when set in fast motion.)

Let \mathcal{LM} denote a process that starts with an event \mathcal{M} and ends with an event \mathcal{L} , and let a number $t^{(F)}(\mathcal{LM})$ denote the elapsed time in an arbitrary frame F, so that

$$t^{(F)}(\mathcal{LM} \cap \mathcal{MN}) = t^{(F)}(\mathcal{LM}) + t^{(F)}(\mathcal{MN}) \quad (10)$$

and $t^{(F)}(\mathcal{LM}) = -t^{(F)}(\mathcal{ML})$.

If in the frame A one has $t^{(A)}(\mathcal{LM}) = t^{(A)}(\mathcal{RS})$ for arbitrary events $\mathcal{L}, \mathcal{M}, \mathcal{R}, \mathcal{S}$, then in the frame B one finds the same equality $t^{(B)}(\mathcal{LM}) = t^{(B)}(\mathcal{RS})$ because the contrary would allow one to judge about the motion of B with respect to A on the basis of internal data in B, in contradiction with the principle of relativity. Further generalization is possible if one exploits Eq. (10) to partition a process into a series of shorter processes over even intervals of time: $t^{(A)}(\mathcal{LM}) = \sigma t^{(A)}(\mathcal{RS})$ entails $t^{(B)}(\mathcal{LM}) = \sigma t^{(B)}(\mathcal{RS})$ for any natural, rational and, finally, real σ . It follows that for any frame F the relation $t^{(F)}(\mathcal{LM})/t^{(F)}(\mathcal{RS})$ shows no dependence on the frame while for any process \mathcal{LM} the relation $t^{(B)}(\mathcal{LM})/t^{(A)}(\mathcal{LM})$ depends only on v defined by Eq. (9).

Let a point body a resting in the frame A and a point body b resting in the frame B meet each other at the event \mathcal{O} and let another freely moving point body g meet a and b at the events \mathcal{A} and \mathcal{B} , respectively. By making use of the laws (5) of free motion, one can get an expression

$$\mathbf{v}_g^{(F)} = \alpha \mathbf{v}_A^{(F)} + (1 - \alpha) \mathbf{v}_B^{(F)} \quad (11)$$

for the velocity $\mathbf{v}_g^{(F)}$ of the body g in the frame F where

$$\alpha = \frac{t^{(F)}(\mathcal{AO})}{t^{(F)}(\mathcal{AO}) - t^{(F)}(\mathcal{BO})}$$

is actually independent of F in accordance with the previous analysis.

Eq. (11) allows one to define a one-parametric family of frames $G[\alpha]$ where g is at rest while the bodies a and b move along the common straight line in opposite directions. Hereinafter this line is referred to as a boost line of a frame $G[\alpha]$ for a given α , and the whole

family $G[\alpha]$ is referred to as a helicoboost class of frames, specified by its two members A and B.

If one considers two different values $\alpha = \alpha_1$ and $\alpha = \alpha_2$ along with two corresponding bodies g_1 and g_2 and applies Eq. (11) to the motion of the body g_2 in the frame $G[\alpha_1]$, one can find that g_2 moves just along the boost line of $G[\alpha_1]$. Since $G[\alpha_1]$ is an arbitrary member of the helicoboost class and g_2 can represent an arbitrary point body (or even a light signal) that moves along the boost line in $G[\alpha_1]$, the boost lines in different frames of one helicoboost class prove equivalent in a sense: Any point bodies that stay (moving or resting) on the boost line in one frame of a given helicoboost class remain (moving or resting) on the boost line in another frame of the same helicoboost class.

If, in addition to the bodies a and b , one chooses more representatives of the frames A and B, one gets more parallel boost lines. To avoid specifying particular boost lines in a reasoning, one can invoke the direction of $\mathbf{v}_B^{(G)}$ or $-\mathbf{v}_A^{(G)}$, which represents a bundle of parallel boost lines in each member G of the helicoboost class. Hereinafter this direction is referred to as a boost direction.

It is possible to define both a boost direction and a helicoboost class of frames without prior reference to its two members: If there are a frame where a set of free point bodies have their velocities parallel or antiparallel to each other, then there are other frames where velocities of the these bodies are also parallel or anti-parallel to each other; an equivalence class of such frames is a helicoboost class; the direction of motion of such free bodies in each frame of the helicoboost class is a boost direction.

B. Spatial transverse effect

The motion of light signals along the boost direction in one frame of a given helicoboost class is an important limiting case of the motions considered in the previous section. Since the light signals represent the propagation of electromagnetic plane wavefronts (7), one can conclude that they propagate along the boost direction in any frame of the helicoboost class. It follows that simultaneous events in a plane perpendicular to the boost direction in one frame appear simultaneous in any other frame of the same helicoboost class, where they also occupy a plane perpendicular to the boost direction.

Let the locations of the above simultaneous events make a certain instantaneous arrange-

ment over the plane wavefront. To represent it as a stationary geometric configuration, one should consider intersections some bundle of boost lines with a dense series of parallel wavefronts which propagate along the boost direction. Since such wavefronts and the boost lines are observable in any frame of the helicoboost class, so is the stationary planar configuration they generate. The possibility of the common geometric configuration shows that the observers in different frames can come to agreement with each other about the orientation of their frames around the boost direction or, in other words, to another equivalence relation between two frames. This relation allows one to partition the helicoboost class into subclasses of identically oriented frames. In the following, a subclass of this kind is referred to as a boost class.

To present the above equivalence relation in terms of a relative position vector, one can write:

$$(\Delta \mathbf{r})_{\perp}^{(A)} \smile (\Delta \mathbf{r})_{\perp}^{(B)}. \quad (12)$$

Here and in the rest of the paper, the notation

$$\left. \begin{aligned} \mathbf{q}^{(A)} &= q_{\parallel}^{(A)} \frac{\mathbf{v}_B^{(A)}}{v} + \mathbf{q}_{\perp}^{(A)}, & q_{\parallel}^{(A)} &\equiv \frac{(\mathbf{q}^{(A)} \cdot \mathbf{v}_B^{(A)})}{v}, \\ \mathbf{q}^{(B)} &= -q_{\parallel}^{(B)} \frac{\mathbf{v}_A^{(B)}}{v} + \mathbf{q}_{\perp}^{(B)}, & -q_{\parallel}^{(B)} &\equiv \frac{(\mathbf{q}^{(B)} \cdot \mathbf{v}_A^{(B)})}{v} \end{aligned} \right\} \quad (13)$$

describes the decompositions of spatial vectors $\mathbf{q}^{(A)}$ and $\mathbf{q}^{(B)}$ in their respective frames.

Since the symbol $(\Delta \mathbf{r})_{\perp}^{(A)}$ in Eq. (12) denotes an arbitrary relative position vector, which connects two arbitrary points in a plane perpendicular to $\mathbf{v}_B^{(A)}$, one can view Eq. (12) as a notation for mapping a geometric configuration in the frame A to that in the frame B. To be accurate and limit oneself by the use of spatial vectors only, without any explicit reference to the concept of Euclidean points, one must define the relation “ \smile ” so that

$$\mathbf{f}_1^{(A)} \smile \mathbf{g}_1^{(B)} \text{ and } \mathbf{f}_2^{(A)} \smile \mathbf{g}_2^{(B)} \text{ entail } \mathbf{f}_1^{(A)} + \mathbf{f}_2^{(A)} \smile \mathbf{g}_1^{(B)} + \mathbf{g}_2^{(B)} \quad (14)$$

and

$$\mathbf{f}_3^{(A)} \smile \mathbf{g}_3^{(B)} \text{ and } \mathbf{f}_4^{(A)} \smile \mathbf{g}_4^{(B)} \text{ entail } (\mathbf{f}_3^{(A)} \cdot \mathbf{f}_4^{(A)}) = (\mathbf{g}_3^{(B)} \cdot \mathbf{g}_4^{(B)}) \quad (15)$$

for any spatial vectors $\mathbf{f}_i^{(A)}$ in the frame A and their counterparts $\mathbf{g}_i^{(B)}$ in the frame B.

C. Spatial longitudinal effect

Suppose in the frame A point bodies P, Q, R, S... are moving with the same velocity $\mathbf{v}_B^{(A)}$ while momentarily (detected as) arranged in a straight line along the boost direction, i.e. parallel to $\mathbf{v}_B^{(A)}$. In the frame B these point bodies are at rest in a straight line along the boost direction. If in the frame A one has $l_B^{(A)}(PQ) = l_B^{(A)}(RS)$, then in the frame B one finds the correspondent equality $l_B(PQ) = l_B(RS)$ due to the relativity principle. Here the notation $l_B^{(A)}(GH)$ refers to a distance between moving bodies G and H momentarily observed in a frame A while $l_B(GH)$ denotes a distance between stationary bodies G and H in a frame B. If one exploits partitioning a line segment in a manner similar to the division (10) of a time interval, one can eventually come to the similar conclusion that $l_B^{(A)}(PQ) = \sigma l_B^{(A)}(RS)$ entails $l_B(PQ) = \sigma l_B(RS)$ for any natural, rational and, finally, real σ . It follows that for any frame F of the boost class the relation $l_B^{(F)}(PQ)/l_B^{(F)}(RS)$ shows no dependence on the frame while for any pair of bodies P and Q stationary in the frame B the relation $l_B(PQ)/l_B^{(A)}(PQ)$ depends only on v .

The application of the above result is not bounded by a comparison of the distances between two bodies in the two frames. In fact, the distance $l_B^{(A)}$ between two moving point bodies in the frame A is a distance between two *simultaneous* events of detecting these bodies in the frame A. In the frame B, these events are not necessarily simultaneous but happening to the same bodies. So the distance l_B between the bodies is a distance between the events, too. Thus, the relation

$$\frac{l_B}{l_B^{(A)}} = K(v) \quad (16)$$

is the same for any pair of events in the straight line along the boost direction, provided that they are simultaneous in the frame A.

Let simultaneous elementary events in the frame A occur over the length l_A in a straight line along the boost direction, i.e. along the direction of motion of the frame B. Physically, they together may be an act of detecting a rigid rod embedded into the frame A. If a flashlight occurs in the middle of the rod, it takes the same time interval $t_A = l_A/2c$ for that light to get to each end of the rod. When instantaneously observed in the frame B, the length of the rod appears to be $l_A^{(B)}$. The arrivals of the above mentioned flashlight at the ends of the moving rod are not simultaneous in the frame B: along the direction of the rod's

motion the time difference makes

$$\Delta t^{(B)} = \frac{l_A^{(B)}/2}{c+v} - \frac{l_A^{(B)}/2}{c-v} = -\gamma^2 \frac{v l_A^{(B)}}{c^2} \quad (17)$$

while the distance between these events is

$$l_B = c \frac{l_A^{(B)}/2}{c+v} + c \frac{l_A^{(B)}/2}{c-v} = \gamma^2 l_A^{(B)} \quad (18)$$

Here and hereafter $\gamma \equiv \gamma(v) \equiv (1 - v^2/c^2)^{-1/2}$.

The equations (17) and (18) take account of the possibility that in different frames a set of the same events may occupy segments of different sizes along their boost direction. Considering the events involved in the inference and the formulation of Eqs. (17) and (18) one can find that $l_A = K(v)l_A^{(B)}$ and $l_B = K(v)l_A$ in accordance with Eq. (16). Then Eq. (18) entails $K(v) = \gamma$, i.e. the length contraction effect

$$l_A = \gamma l_A^{(B)} \quad (19)$$

D. Time dilation effect

To get to another well known effect, one can exploit such thought construct as the light clock³¹, where a light pulse propagates back and forth between two mirrors held parallel and apart at the fixed distance l_0 in their rest frame. In that frame, the line of the light propagation is perpendicular to the mirrors and can be referred to as the axis of the clock. The proper duration of the clock's cycle, i.e. the round-trip time of the light pulse in the clock's rest frame, makes

$$\Delta\tau = \frac{2l_0}{c}. \quad (20)$$

Let an observer be moving with a speed v in the light clock's rest frame and along the axis of the light clock. The observer can detect the contracted length $l = l_0/\gamma$ of the light clock and find that the clock's cycle takes

$$\Delta t = \frac{l}{c-v} + \frac{l}{c+v} = 2\gamma^2 l/c = 2\gamma l_0/c. \quad (21)$$

Here it is worth remarking that the light clock is an axiliary dedicated thought construct, which can in no way include the actual reflections of an electromagnetic wave from substance

of the mirrors but embodies the properly collated sequence of events only. This means that Eq. (20) and Eq. (21) together entail the time dilation effect

$$\Delta t = \gamma \Delta \tau, \quad (22)$$

which gives the laboratory time interval Δt between two events at a moving point during its given proper time interval $\Delta \tau$.

To show relations between the spatiotemporal effects of the relativity theory in one's practice of teaching, one may also address the following inference.

Let the light clock of proper length l_0 rest in the frame A so that the axis of the clock is perpendicular to the direction of motion of the frame B, i.e. to the boost direction. In the frame B, the axis of the clock is also perpendicular to the boost direction, and the main cycle of the clock corresponds to the light signal path shown in Fig. 1. Here l is a length

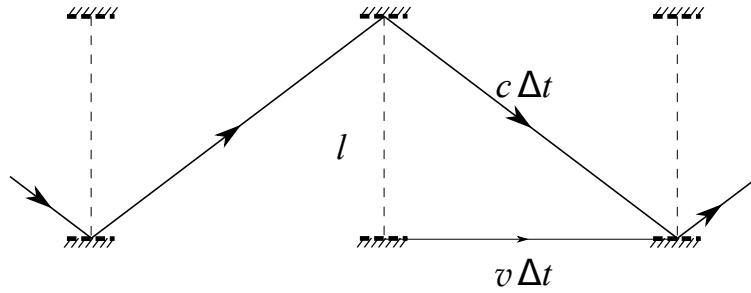


FIG. 1. The path of the light pulse in the transversally moving light clock.

of the clock simultaneously observed in the frame B as being in the plane perpendicular to the direction of the clock's motion. In accordance with Fig. 1 the duration Δt of the clock's main cycle in the frame B should obey the equation

$$l^2 + (v\Delta t/2)^2 = (c\Delta t/2)^2.$$

This is consistent with the time dilation effect (22) and the proper duration (20) of the clock's main cycle because the spatial transversal effect (12) yields $l = l_0$ when the motion of the light clock and the propagation of the light pulse occur in the same plane.

E. Retardation effect

Eq. (12) suggests that one can generalize Eq. (19) and Eq. (17): If in the frame A simultaneous events occur in the transversal spatial slab $\Delta r_{\parallel}^{(A)}$ thick, then in the frame B

they occur within the slab

$$\Delta r_{\parallel}^{(B)} = \gamma \Delta r_{\parallel}^{(A)} \quad (23)$$

and with the time spread

$$\Delta t^{(B)} = -\gamma \frac{v \Delta r_{\parallel}^{(A)}}{c^2} = -\gamma \frac{(\mathbf{v}_B^{(A)} \cdot \Delta \mathbf{r}^{(A)})}{c^2}. \quad (24)$$

IV. RELATION BETWEEN TWO FRAMES

To describe the relation between the frames A and B in terms of the position vectors $\mathbf{r}^{(A)}$ and $\mathbf{r}^{(B)}$ and the time moments $t^{(A)}$ and $t^{(B)} = 0$ of a given arbitrary event \mathcal{E} one should first of all identify the place of a common reference event \mathcal{O} with the origins $\mathbf{r}^{(A)} = 0$ and $\mathbf{r}^{(B)} = 0$ of the frames A and B as well as its time moment with the readings $t^{(A)} = 0$ and $t^{(B)} = 0$ of their clocks.

Let two elementary events occur in the frame A at a given time moment $t^{(A)}$: an event \mathcal{O}^A occurs at the origin $\mathbf{r}^{(A)} = 0$ while \mathcal{E} occurs at $\mathbf{r}^{(A)} \neq 0$. Due to the time dilation effect, the moment of observing \mathcal{O}^A in the frame B equals to the reading $\gamma t^{(A)}$ of a clock fixed at the origin $\mathbf{r}^{(B)} = 0$. Due to the time spread effect (24), in the frame B the difference in time between \mathcal{E} and \mathcal{O}^A

$$\Delta t^{(B)} = -\gamma \frac{v r_{\parallel}^{(A)}}{c^2}.$$

Therefore, the time moment of observing \mathcal{E} in the frame B

$$t^{(B)} = \gamma t^{(A)} - \gamma \frac{v r_{\parallel}^{(A)}}{c^2}. \quad (25)$$

Further, if in the frame B the plane Π_B is perpendicular to $\mathbf{v}_B^{(A)}$ and keeps passing through the origin $\mathbf{r}^{(B)} = 0$, then, in accordance with the analysis in Section III A, in the frame A it forms the plane $\Pi_B^{(A)}$ perpendicular to $\mathbf{v}_A^{(B)}$ and moving with this velocity. Evidently, the distance from the event \mathcal{E} to the plane $\Pi_B^{(A)}$ makes $\Delta r_{\parallel}^{(A)} = r_{\parallel}^{(A)} - v t^{(A)}$. If $\mathbf{r}^{(B)}$ denotes the position of \mathcal{E} in the frame B, then the length contraction effect (23) helps us conclude that in the frame B the distance from the event \mathcal{E} to the plane Π_B is

$$r_{\parallel}^{(B)} = \gamma \left(r_{\parallel}^{(A)} - v t^{(A)} \right) \quad (26)$$

while Eq. (12) yields

$$\mathbf{r}_{\perp}^{(A)} \sim \mathbf{r}_{\perp}^{(B)}. \quad (27)$$

Within the described coordinate-free approach, the relationships (25), (26) and (27) can be view as transformation rules. To show the dependence on the frames A and B explicitly one should only decipher the notation introduced by Eq. (9) and Eq. (13):

$$t^{(B)} = \gamma_{AB} \left[t^{(A)} - \frac{(\mathbf{v}_B^{(A)} \cdot \mathbf{r}^{(A)})}{c^2} \right], \quad (28)$$

$$-\frac{(\mathbf{r}^{(B)} \cdot \mathbf{v}_A^{(B)})}{v_{AB}} = \gamma_{AB} \left[\frac{(\mathbf{r}^{(A)} \cdot \mathbf{v}_B^{(A)})}{v_{AB}} - v_{AB} t^{(A)} \right], \quad (29)$$

$$\mathbf{r}^{(B)} - \frac{(\mathbf{r}^{(B)} \cdot \mathbf{v}_A^{(B)})}{v_{AB}^2} \mathbf{v}_A^{(B)} \sim \mathbf{r}^{(A)} - \frac{(\mathbf{r}^{(A)} \cdot \mathbf{v}_B^{(A)})}{v_{AB}^2} \mathbf{v}_B^{(A)}. \quad (30)$$

In a more customary form, the transformation rules (28)-(30) can also be presented as the mapping

$$\begin{pmatrix} ct^{(B)} \\ \mathbf{r}^{(B)} \end{pmatrix} \leftrightarrow \mathbf{M}_{(A)}^{(B)} \odot \begin{pmatrix} ct^{(A)} \\ \mathbf{r}^{(A)} \end{pmatrix} \quad (31)$$

where

$$\mathbf{M}_{(A)}^{(B)} \equiv \begin{pmatrix} \gamma_{AB} & -\gamma_{AB} \mathbf{v}_B^{(A)}/c \\ \gamma_{AB} \mathbf{v}_A^{(B)}/c & 1 - \gamma_{AB} \mathbf{v}_A^{(B)} \otimes \mathbf{v}_B^{(A)}/v_{AB}^2 \end{pmatrix}, \quad (32)$$

where the symbol \leftrightarrow unites the meaning of $=$ and the meaning of \sim while the symbol \odot unites the meaning of the usual product of two numbers and the meaning of the dot product of two spatial vectors; the symbol \otimes denotes the dyadic (outer) product.

Historically, the early attempts to obtain such a mapping did not lead to a correct/unambiguous expression of $\mathbf{r}^{(B)}$ via $\mathbf{r}^{(A)}$ because of the failure to distinguish between a column vector and a true vector.³³ To keep being mathematically correct the later treatment³⁴ could not avoid resorting to coordinates but appeared limited to boost transformations (see the final remark in Section V B.)

V. COORDINATE TRANSFORMATIONS IN THE RELATIVITY THEORY

A. Lorentz transformation

As soon as one specifies coordinate systems in the frames A and B, in terms of relations between their unit base vectors $\{\mathbf{e}_\alpha^{(A)}\}$ and $\{\mathbf{e}_\alpha^{(B)}\}$ themselves as well as to some physical

directions, one can arrive at the relationship

$$\vec{\rho}^{(B)} = \mathbb{M}_A^B \vec{\rho}^{(A)} \quad (33)$$

between the column vectors

$$\vec{\rho}^{(A)} = \begin{pmatrix} ct^{(A)} \\ x^{(A)} \\ y^{(A)} \\ z^{(A)} \end{pmatrix}, \quad \vec{\rho}^{(B)} = \begin{pmatrix} ct^{(B)} \\ x^{(B)} \\ y^{(B)} \\ z^{(B)} \end{pmatrix},$$

made of time and spatial coordinates of a given event as observed in the inertial coordinate systems A and B, respectively. The choice

$$\mathbf{e}_x^{(A)} = \frac{\mathbf{v}_B^{(A)}}{v_{AB}}, \quad \mathbf{e}_x^{(B)} = -\frac{\mathbf{v}_A^{(B)}}{v_{AB}}, \quad \mathbf{e}_y^{(A)} \smile \mathbf{e}_y^{(B)}, \mathbf{e}_z^{(A)} \smile \mathbf{e}_z^{(B)}$$

for the unit base vectors turns Eqs. (28)-(30) into Eq. (33) with $\mathbb{M}_A^B = \mathbb{L}(v_{AB}\vec{e}_x)$ where

$$\mathbb{L}(v\vec{e}_x) = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{e}_x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (34)$$

i.e. the matrix of the original form of Lorentz transformation.

B. Boost transformations in the physics literature

Boosts make a well known class of the transformations (33) introduced in graduate level physics courses with an aid of its matrix

$$\mathbb{M}_A^B = \mathbb{L}(\vec{v}_B^{(A)}) \quad (35)$$

where³⁷

$$\mathbb{L}(v\vec{n}) = \begin{pmatrix} \gamma & -\gamma v n_x/c & -\gamma v n_y/c & -\gamma v n_z/c \\ -\gamma v n_x/c & 1 + (\gamma - 1)n_x^2 & (\gamma - 1)n_x n_y & (\gamma - 1)n_x n_z \\ -\gamma v n_y/c & (\gamma - 1)n_y n_x & 1 + (\gamma - 1)n_y^2 & (\gamma - 1)n_y n_z \\ -\gamma v n_z/c & (\gamma - 1)n_z n_x & (\gamma - 1)n_z n_y & 1 + (\gamma - 1)n_z^2 \end{pmatrix}, \quad \vec{n} = \begin{pmatrix} 0 \\ n_x \\ n_y \\ n_z \end{pmatrix}, \quad (36)$$

$n_x^2 + n_y^2 + n_z^2 = 1$, the elements of the column vector $\vec{v}_B^{(A)}$ are the A components of the velocity of B with respect to A.

Due to the identities³⁹

$$\mathbb{L}(v\vec{n}) = \mathbb{R}^{-1}\mathbb{L}(v\vec{e}_x)\mathbb{R}, \quad \vec{n} = \mathbb{R}\vec{e}_x \quad (37)$$

with the rotation matrix

$$\mathbb{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x & -n_y & -n_z \\ 0 & n_y & 1 - \frac{n_y^2}{1+n_x} & -\frac{n_y n_z}{1+n_x} \\ 0 & n_z & -\frac{n_z n_y}{1+n_x} & 1 - \frac{n_z^2}{1+n_x} \end{pmatrix},$$

one can say that $\mathbb{L}(\vec{v})$ incorporates \vec{v} as an observable direction.

It is easy to find that textbooks' authors exploit boost transformations them only to show a derivation of the Thomas precession contribution to the spin motion of a relativistically moving charged particle.³⁵ However, despite this seemingly definite connection to observable phenomena, in all the texts up till now the presentations that facilitate algebraic manipulations of boosts are favored over those which would treat a boost as a physically defined operation/motion or relations between physical objects.

The matrices $\mathbb{L}(\vec{v})$ parametrized by various \vec{v} belong to the grouplike structure formed by general transformation matrices $\mathbb{M} = \mathbb{R}'\mathbb{L}(v\vec{e}_x)\mathbb{R}''$ where \mathbb{R}' and \mathbb{R}'' represent two arbitrary rotations. The algebraic properties of that structure were the subject of extensive investigation, such as the treatment in Ref. 40, but could not provide any better understanding of the spin precession or anything else because the physical meaning of $\mathbb{L}(\vec{v})$ itself remained out of consideration.

The only benefit of such an analysis for physicists seems to be that it has found no reason to consider a boost as the Lorentz transformation “without rotation,” in attempt to generalize the idea of parallel transport, such as the definition (ii) at p. 871 in Ref. 38. As a result, to “a boost transformation” mathematicians prefer to apply cautious terms such as “an aligned axis Lorentz transformation” (see p. 236 in Ref. 41.)

The most visible manifestation of the problem is that no text attempts to formulate a physics based definition of a boost, thereby preventing any reasonable use of that concept. The reference formula (36) makes it difficult to get an idea how mutual orientations of several

bodies change when set in motion, because (36) alone provides no hints how to choose the axes of the coordinate systems in use. The seemingly key formula (37) appears to be nothing but a relation between two representations (36) and (34) for one relative motion, of which the formal simplicity of (34) may even be misleading:

When asked about the direction of the $x^{(A)}$ -axis, someone may correctly infer from (34) that it is the direction of the $x^{(B)}$ -axis instantaneously observed in the coordinate system A. But the problem is that a researcher must be able to identify any directions before establishing/verifying (theoretically/experimentally) the relationships (34). To avoid that apparent logical circle, one might take the above description of the direction as a definition and an explicit starting point in a derivation of the transformation rule (34). Needless to say that such an accurate approach can hardly be found in the existing, history-oriented, presentations of the relativity theory.

Now the coordinate-free description for the relation between two arbitrary frames in Section IV allows one to formulate logically consistent and physically explicit definition of a boost coordinate transformation.

C. Definition of a boost transformation and derivation of its matrix

Aside from the origins, the above consideration refers to no elements of coordinate systems. To set up Cartesian coordinate systems in the frames A and B one should specify the direction of their spatial base unit vectors $\mathbf{e}_{A\alpha}$ and $\mathbf{e}_{B\alpha}$ for $\alpha = x, y, z$. Let

$$\mathbf{e}_{A\alpha\perp} \smile \mathbf{e}_{B\alpha\perp} \quad (38)$$

and

$$\left(\mathbf{e}_{A\alpha} \cdot \mathbf{v}_B^{(A)} \right) = - \left(\mathbf{e}_{B\alpha} \cdot \mathbf{v}_A^{(B)} \right) \equiv v_\alpha. \quad (39)$$

In other words, in view of Eq. (15), $-\mathbf{v}_A^{(B)}$ makes the same angles with $\mathbf{e}_{B\alpha}$ in the frame B as $\mathbf{v}_B^{(A)}$ does with $\mathbf{e}_{A\alpha}$ in the frame A. Then, by definition, the boost is the transformation (25) of time along with the transformation of coordinates of an event between the coordinate systems satisfying the conditions (38) and (39).

The equivalence relations (38) and (12) along with the property (15) yield the equality of numbers

$$\left(\mathbf{r}_\perp^{(A)} \cdot \mathbf{e}_{A\alpha\perp} \right) = \left(\mathbf{r}_\perp^{(B)} \cdot \mathbf{e}_{B\alpha\perp} \right). \quad (40)$$

Eqs. (39) and (26) allows one to rewrite the dot products as

$$\begin{aligned} \left(\mathbf{r}_{\perp}^{(A)} \cdot \mathbf{e}_{A\alpha\perp} \right) &= \left(\mathbf{r}_{\perp}^{(A)} \cdot \mathbf{e}_{A\alpha} \right) = \left(\left(\mathbf{r}^{(A)} - r_{\parallel}^{(A)} \frac{\mathbf{v}_B^{(A)}}{v} \right) \cdot \mathbf{e}_{A\alpha} \right) = \\ &= r_{\alpha}^{(A)} - r_{\parallel}^{(A)} \frac{v_{\alpha}}{v} = r_{\alpha}^{(A)} - \frac{(\mathbf{v}_B^{(A)} \cdot \mathbf{r}^{(A)})}{v} \frac{v_{\alpha}}{v} = r_{\alpha}^{(A)} - \frac{v_{\alpha} v_{\beta}}{v^2} r_{\beta}^{(A)} \end{aligned}$$

and

$$\begin{aligned} \left(\mathbf{r}_{\perp}^{(B)} \cdot \mathbf{e}_{B\alpha\perp} \right) &= \left(\mathbf{r}_{\perp}^{(B)} \cdot \mathbf{e}_{B\alpha} \right) = \left(\left(\mathbf{r}^{(B)} + r_{\parallel}^{(B)} \frac{\mathbf{v}_A^{(B)}}{v} \right) \cdot \mathbf{e}_{B\alpha} \right) = \\ &= r_{\alpha}^{(B)} - r_{\parallel}^{(B)} \frac{v_{\alpha}}{v} = r_{\alpha}^{(B)} - \gamma (r_{\parallel}^{(A)} - vt^{(A)}) \frac{v_{\alpha}}{v} = \\ &= r_{\alpha}^{(B)} + \gamma v_{\alpha} t^{(A)} - \gamma \frac{(\mathbf{v}_B^{(A)} \cdot \mathbf{r}^{(A)})}{v} \frac{v_{\alpha}}{v} = r_{\alpha}^{(B)} + \gamma v_{\alpha} t^{(A)} - \gamma \frac{v_{\alpha} v_{\beta}}{v^2} r_{\beta}^{(A)} \end{aligned}$$

Here the common pithy notation is used: $r_x \equiv x, r_y \equiv y, r_z \equiv z$ while the repeated index β implies the summation over all its values. Then Eq. (40) entails

$$r_{\alpha}^{(B)} = r_{\alpha}^{(A)} - \gamma v_{\alpha} t^{(A)} + (\gamma - 1) \frac{v_{\alpha} v_{\beta}}{v^2} r_{\beta}^{(A)},$$

which is just the spatial part of the transformation (33) with the matrix (35).

In terms of column vectors the above equation can be written as

$$\vec{r}^{(B)} = \vec{r}^{(A)} - \gamma \vec{v}_B^{(A)} t^{(A)} + (\gamma - 1) \frac{(\vec{r}^{(A)} \cdot \vec{v}_B^{(A)}) \vec{v}_B^{(A)}}{v^2}. \quad (41)$$

This form of a boost transformation was obtained from Eq. (37) in Ref. 34.

D. Boost operation in the laboratory frame

One can address the definition of a boost in the previous section so as to come to physically meaningful conclusions directly. An important example is a simple analytical description for the distortion that a Cartesian coordinate basis exhibits while simultaneously observed in the frame where it experiences a boost operation.

Fig. 2 presents a plane parallel to a spatial vector \mathbf{q}_A and the velocity $\mathbf{v}_B^{(A)}$ of the frame B, both the vectors being sets of simultaneous events in the frame A. The boost operation applied to the vector

$$\mathbf{q}_A = q_{\parallel} \frac{\mathbf{v}_B^{(A)}}{v} + \mathbf{q}_{A\perp}$$

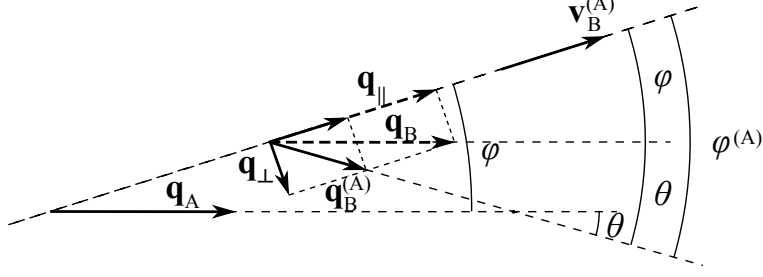


FIG. 2. The moving vector \mathbf{q}_B is instantaneously observed as the vector $\mathbf{q}_B^{(A)}$.

results in the vector

$$\mathbf{q}_B = -q_{\parallel} \frac{\mathbf{v}_A^{(B)}}{v} + \mathbf{q}_{B\perp}, \quad \mathbf{q}_{B\perp} \sim \mathbf{q}_{A\perp}$$

which is spatial in the frame B and parallel to the above plane, too. Due to the definition of the boost and the length contraction effect, in the frame A the vector \mathbf{q}_B is perceived (instantaneously observed) as the vector

$$\mathbf{q}_B^{(A)} = (q_{\parallel}/\gamma) \frac{\mathbf{v}_B^{(A)}}{v} + \mathbf{q}_{\perp}.$$

Since

$$\tan \varphi = \frac{|\mathbf{q}_{\perp}|}{q_{\parallel}}$$

for the angle φ between \mathbf{q}_A and $\mathbf{v}_B^{(A)}$ as well as between \mathbf{q}_B and $-\mathbf{v}_A^{(B)}$, and

$$\tan \varphi^{(A)} = \gamma \frac{|\mathbf{q}_{\perp}|}{q_{\parallel}}$$

for the angle $\varphi^{(A)}$ between $\mathbf{q}_B^{(A)}$ and $\mathbf{v}_B^{(A)}$, one can find that

$$\tan \theta = \frac{(\gamma - 1) \tan \varphi}{1 + \gamma \tan^2 \varphi}$$

for the angle $\theta = \varphi^{(A)} - \varphi$ between $\mathbf{q}_B^{(A)}$ and \mathbf{q}_A (see Fig. 2.)

Sometimes it may be reasonable to change the laboratory coordinate system with the aid of the boost transformation so as to reach simpler state of motion, to reduce the form of interaction law etc. Then, in the limit $v \ll c$, the angle $\theta = O(v^2/c^2)$ shows what relative error is not properly compensated in case one fails to transform all relevant quantities appropriately.

VI. CONCLUSION

A boost direction for a pair of inertial reference frames is that spatial direction in one of the frames along which the other frame moves. Free motions of point particles make an instrumentation for identifying the boost direction as well as events on a straight line along that direction. The concept of a boost direction secures the formulation of the basic relativity effects in a physics-based manner, which, eventually, results in the relation between two arbitrary frames in terms of their position vectors and time moments for a given event.

Within the physics-based approach, addressing the transformation of coordinates implies specifying some coordinate systems in each frame in terms of physical objects/directions. This yields a logically consistent and physically meaningful presentation of the coordinate transformations commonly exploited in the special relativity theory, which makes observable effects associated with those transformations evident. In particular, for a Cartesian coordinate system subjected to a boost coordinate transformation, the coordinate-free technique of reasoning allows one to evaluate its instantaneously observable apparent distortion easily.

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¹ A. Einstein “Zur Elektrodynamik der bewegter Körper,” Ann. Phys. **17**, 891-921 (1905)

² See, e.g., chs. I-VI in Ref. 3, chs. I-IV in Ref. 4, chs. I-IV in Ref. 5, pt. I in Ref. 6, chs. IV-VI in Ref. 7.

³ L. Silberstein, *The Theory of Relativity*, (Macmillan and Co., 1914)

⁴ Richard C. Tolman, *The Theory of Relativity of Motion*, (University of California Press, 1917)

⁵ Robert D. Carmichael, *The Theory of Relativity*, 2nd edition (Wiley, 1920)

⁶ W. Pauli, *The Theory of Relativity*, (Pergamon Press, 1958)

⁷ Max Born, *Einstein's Theory of Relativity*, (Dover Publications, 1962)

⁸ See, e.g., chs. I-IV in Ref. 9, chs. I-II in Ref. 10, ch. 1 in Ref. 11, Sections 11.1-11.4 in Ref. 12, Sections 7.1-7.3 in Ref. 13.

⁹ Peter G. Bergmann, *Introduction to the Theory of Relativity*, (Dover Publications, 1976)

¹⁰ C. Møller, *The Theory of Relativity*, (Oxford University Press, 1955)

¹¹ Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics*, (W. H. Freeman and Co.,

1992)

- ¹² John David Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, 1998)
- ¹³ Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics*, 3rd edition (Addison-Wesley, 2000)
- ¹⁴ V. Fock, *The Theory of Space Time and Gravitation*, 2nd Revised Edition (Pergamon Press, 1964)
- ¹⁵ R. Hagedorn, *Relativistic Kinematics. A Guide to Kinematic Problems of High-energy Physics*, (W. A. Benjamin, Inc., 1964)
- ¹⁶ Since both electromagnetism and gravity manifest themselves via the concept of force, they allow one to distinguish directions and thereby support projective geometry at least. The next step should involve point particles, i.e. such entities that can strongly interact over a sufficiently short range. Then the application of first Newton's law to several particles on the same straight line helps one identify stationary particles, which can embody Euclidean points. It enables one to introduce the concept of length in the usual manner.
- The discussed approach implies that the physical regularities involved can be presented in pre-Euclidean/prenumeric terms to describe experiments/observations more directly. To get an idea about a technique of such description, see Sections V and VI in Ref. 17.
- ¹⁷ Serge A. Wagner, "How to introduce physical quantities physically," <<http://arxiv.org/pdf/1506.04122v1>>
- ¹⁸ For example, if someone would choose to store information about a local direction in the angular momentum of a hydrogen atom, he will inevitably fail both due to the well known unavoidable quantization of that quantity and because of the small lifetime of the corresponding excited state of motion. In accordance to the simple consideration known as Bohr's theory of a hydrogen atom, it occurs over the spatial scale, called the Bohr radius and recognized as a characteristic atomic scale.

In general, the elementary charge e , Plank's constant \hbar , electron's rest mass m_e , proton's rest mass m_p are well known magnitudes which reveal the quantization of electric charge, angular momentum and mass/energy, respectively. The dimensional and/or provisional theoretical analysis can yield a set of characteristic spatial scales

$$f(e^2/\hbar c, m_e/m_p, \dots) \hbar/m_e c$$

where $f(\alpha, \xi, \dots)$ is a dimensionless function of the dimensionless quantities α, ξ, \dots . The cases $f = \alpha$, $f = 1$ and $f = \alpha^{-1}$ yield the well known scales: the classical electron radius (in fact, the characteristic size of proton/neutron), Compton wavelength of electron and the Bohr radius. Here the use of the CGS system of units is assumed, so one can take c as a parameter in Maxwell equations, not necessarily the observable speed of light.

Though no formal analysis of applicability of Euclidean geometry has yet been published, it is over scales less than Compton wavelength of electron where some authors have found it difficult to provide a consistent concept of a spatial coordinate. See p. 40 in Ref. 19 and p. 2 in Ref. 20 for simple order-of-magnitude considerations and, e.g., Ref. 21 for the exemplars of existing theoretical approach to the problem.

- ¹⁹ James D. Bjorken, Sidney D. Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill, 1964)
- ²⁰ Berestetskii, E. M. Lifshitz and L. P. Pitaevskii *Quantum Electrodynamics. Course of Theoretical Physics, vol. 4*, 2nd edition, (Pergamon Press 1982)
- ²¹ G. C. Hegerfeldt, “Instantaneous spreading and Einstein causality in quantum theory,” *Ann. d. Phys.* **7**, 716-725 (1998)
- ²² The first more or less general formulation of the relativity principle is believed to belong H. Poincaré. See, e.g., p. 111 in Ref. 23 or p. 107 and p. 300 in Ref. 24
- ²³ H. Poincaré, *Science and Hypothesis* (London and Newcastle-on-Cyne: The Walter Scott Publishing Co., 1905)
- ²⁴ H. Poincaré, *The Foundation of Science* (NY: The Science Press New York and Garrison, 1913)
- ²⁵ The thought that first Newton’s law has the same form in all inertial frames was viewed as trivial and, for this reason, exploited implicitly (until the accurate presentation in Ref. 14 has appeared.) In contrast, Einstein’s idea that electromagnetic phenomena, such as the propagation of a spherical wave, was apparently perceived as nontrivial. So the statement that a spherical wavefront keeps its form in all frames appeared to be a popular explicit premise for the derivation of Lorentz transformation; see, e.g., p. 100 in Ref. 3 and p. 9 in Ref. 6.
- ²⁶ This is just the condition in the equations (8.05) and (8.06) at p. 21 in Ref. 14. Interestingly, Fock starts his consideration with the general wavefront equation, which, in principle, can describe non-spherical wavefronts, e.g., from two interfering point sources. But eventually he has narrowed his inference with no explicit reasoning.
- ²⁷ See the flat spacetime limit for “the remark” (e) at p. 69 followed by the reasoning at p. 82 in

- Ref. 28 or the description of “the second clock effect” at p. 122 in Ref. 29.
- ²⁸ J. Ehlers, F. A. E. Pirani, A. Schild, “The Geometry of Free Fall and Light Propagation,” in *General relativity, papers honour of J. L. Synge*, edited by L. O’Raifeartaigh (Clarendon Press, Oxford, 1972), p. 63–84.
- ²⁹ J. Ehlers, A. Schild, “Geometry in a Manifold with Projective Structure,” *Commun. Math. Phys.* **32**, 119–146 (1973)
- ³⁰ This is an implicit reason why Fock¹⁴ starts his inference of a boost with a generally nonlinear transformation of the space and time variables and addresses some concepts of Riemannian geometry as elements of a suitable notation.
- ³¹ The first description of the light clock is given at p. 293 in Ref. 32.
- ³² Max von Laue, “Die Nordströmsche Gravitationstheorie. (Bericht.),” *Jahrbuch der Radioaktivität und Elektronik* **14**, 264-312 (1917)
- ³³ See the footnote 24 at pp. 10 and 11 in Ref. 6 and the solution to Problem 1-3 at pp. 8 and 9 in Ref. 15 for the examples of such reasoning in educational texts for physicists.
- ³⁴ James T. Cushing, “Vector Lorentz Transformations,” *Am. J. Phys.* **35**, 858-862 (1967)
- ³⁵ Cf. Section 7.3 in Ref. 13 , Sections 11.7 and 11.8 in Ref. 12, §41.4 in Ref. 36.
- ³⁶ Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, (Freeman, 1973)
- ³⁷ See, e.g., Eq. (11.98) at p. 547 in Ref. 12 or Eq. (24) at p. 872 in Ref. 38.
- ³⁸ Charles P. Frahm, “Representing arbitrary boosts for undergraduates,” *Am. J. Phys.* **47** (10), 870-872 (1979)
- ³⁹ See, e.g., Eq. (13) in Ref. 34.
- ⁴⁰ Abraham A. Ungar, *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession*, (Kluwer Academic Publishers, 2002)
- ⁴¹ W. L. Kennedy, “Thomas rotation: a Lorentz matrix approach,” *Eur. J. Phys.* **23**, 235-247 (2002)